We consider numerical solving the Cauchy problem for backward heat equation via the generalized method of Lie-algebraic discrete approximations. Discretization of the equation is performed by all variables in equation leading to a factorial rate of convergence in the case of quasi representations for differential operator are built by means of Lagrange interpolation. The rank of a finite dimensional operator is determined and approximation properties are investigated. Error estimations and the factorial rate of convergence are proven.

Ключові слова: generalized method of Lie algebraic discrete approximations, backward heat equation, finite dimensional quasi representation, Lagrange’s polynomial, factorial convergence.

1. INTRODUCTION

Backward heat equation has applications in different fields: image processing, signal processing, eliminating of diffusion. Hence effective numerical solution is an important problem besides the variety of different approaches [16]. In this paper we propose a numerical scheme built via generalized method of Lie-algebraic discrete approximations for backward heat equation [20] and prove the factorial rate of convergence of numerical scheme.

The generalized method of Lie algebraic method was introduced in [3] and developed in [4, 18, 19], which is based on classic Lie algebraic method of discrete approximations. The history of classical approach, open questions in this field and further development guidelines are analyzed in [1]. Key findings of classical approach may be found in [6, 7, 9 – 16, 21, 22].

The main problem analyzed in these papers is the Cauchy problem for evolution equation which is considered in a bounded domain $\Omega \subseteq (a, b) \subset R$ with time limit $T < +\infty$ and cylinder $Q_T = \Omega \times (0, T)$:

\[
\begin{align*}
\text{find function } u &= u(x_1, x_2, ..., x_n, t) \text{ such that} \\
\frac{\partial u}{\partial t} &= K(t, x)u + f(x, t), \ x \in \Omega \subset R^n, \ t > 0, \\
\left. u \right|_{t=0} &= \varphi(x) \in B, \\
\end{align*}
\]

where $B$ denotes some functional Banach space, linear operator $K$ is assumed to be a formal polynomial of elements from the Lie algebra \( \{ x, \partial / \partial x, 1 \} \) and can be represented as

\[
K = a_k \frac{\partial^k}{\partial x^k} + a_{k+1} \frac{\partial^{k+1}}{\partial x^{k+1}} + \ldots + a_{k+r} \frac{\partial^{k+r}}{\partial x^{k+r}},
\]
where \( a_{k,i} \in R \) for all \( i = 0, p \) and \( a_k \neq 0 \) and \( k \geq 1 \). Similarly, as in Calogero’s method, the Heisenberg-Weyl algebra \( G = \bigoplus_{j=1}^{q} \langle x_j, \partial / \partial x_j, 1 \rangle \) has been used as a basic algebraic tool for constructing the corresponding discrete approximations \( X_{j}^{(n)}, Z_{j}^{(n)}, I_{j}^{(n)} \in \bigotimes_{j=1}^{q} R^n \).

Using \( q \)-dimensional Lagrange interpolation scheme problem (1) is reduced to the Cauchy problem in the following form:

\[
\begin{align*}
\text{find function } & u_{(a)} = u_{(a)}(t) \text{ such that } \\
\frac{du_{(a)}}{dt} &= K_{(a)}u_{(a)} + f_{(a)}, \quad t > 0, \\
u_{(a)}\big|_{t=0} &= \varphi_{(a)} \in B_{(a)}
\end{align*}
\]

(2)

where \( K_{(a)} \) denotes finite dimensional quasi representation of differential operator \( K \), and \( B_{(a)} \) denotes finite dimensional space of approximations. System (2) is solved by means of Euler’s or Runge-Kutta’s method [1, 7, 9, 12].

Since reduced problem (2) is solved making use of some numerical algorithm the rate of time convergence is constrained by the convergence rate of the method based on, hence Lie-algebraic discrete approximations for spacial variables rate of convergence is factorial [7, 9].

This restriction led to development of the Generalized Method of Lie-algebraic discrete approximations proposed in [3], convergence rate for the time variable becomes factorial [18].

The main idea of Generalized Method of Lie-algebraic discrete approximations is the following. We take the Banach spaces \( V = C^{2+j,p-1}_{x,t}(Q_{r}) \cap C(\bar{Q}_{r}), C = C(\bar{Q}_{r}) \) and formulate the Cauchy problem

\[
\begin{align*}
\text{find function } & u = u(x,t) \in V \text{ such that } \\
u &= Ku + f, \quad \forall (x,t) \in Q_{r}, \\
u\big|_{t=0} &= \varphi \in V,
\end{align*}
\]

(3)

where \( \varphi = \varphi(x) \in V \) denotes initial conditions, \( f = f(x,t) \in C \) represents internal sources and \( K \) denotes the differential operator of a problem.

According to [3, 4] we introduce substitution \( u(x,t) = v(x,t) + \varphi(x) \) into (3) which leads to considering the auxiliary Cauchy problem with homogeneous initial condition

\[
\begin{align*}
\text{find function } & v = v(x,t) \in V \text{ such that } \\
v &= Kv + K\varphi + f, \quad \forall (x,t) \in Q_{r}, \\
v\big|_{t=0} &= 0.
\end{align*}
\]

(4)

The idea of such substitution is to reduce the computation effort with preserved accuracy which was demonstrated in the case of boundary value problem for elliptic equation [2].

The solution of problem (4) we seek in the subspace of such functions which are homogeneous at the initial moment of time: \( B = \{ v \in V : v\big|_{t=0} = 0 \} \).
Denoting the structure elements in (4) by
we obtain a problem for operator equation:
\[
\begin{aligned}
\text{for given operator } A : B \to C \text{ and element } \tilde{f} \in C, \\
\text{find element } v \in B \text{ such that } Av = \tilde{f}.
\end{aligned}
\]

The Cauchy problem was reduced into the problem for the operator equation. This operator equation was solved by means of the generalized method of Lie algebraic discrete approximations and there is proven its convergence for some particular cases [3, 18, 19]. We focus on application of the scheme to Cauchy problem for backward heat equation in this paper.

2. PROBLEM FORMULATION

Considering a bounded domain \( \Omega := (a, b) \subset \mathbb{R} \), time limit \( T < +\infty \), cylinder \( Q_T = \Omega \times (0, T] \) we take the Banach spaces \( V = C^{2,1}(Q_T) \cap C(\overline{Q_T}) \), \( C = C(Q_T) \) and formulate the Cauchy problem
\[
\begin{aligned}
\text{find function } u = u(x,t) \in V \text{ such that } \\
u_t = -a u_{xx} + f, \quad \forall (x,t) \in Q_T, \\
u|_{t=0} = \varphi \in V,
\end{aligned}
\]
where \( a > 0 \) denotes heat transition parameter, \( \varphi = \varphi(x) \in V \) denotes initial conditions and \( f = f(x,t) \in C \) represents internal sources. This problem is equivalent to
\[
\begin{aligned}
\text{find function } u = u(x,t) \in V \text{ such that } \\
u_t = au_{xx} + f, \quad \forall (x,t) \in Q_T, \\
u|_{t=0} = \varphi \in V,
\end{aligned}
\]

According to [3, 4] we introduce substitution \( u(x,t) = v(x,t) + \varphi(x) \) into (5) which leads to considering the auxiliary Cauchy problem with homogeneous initial condition
\[
\begin{aligned}
\text{find function } v = v(x,t) \in V \text{ such that } \\
v_t = -av_{xx} - a\varphi^*_{xx} + f, \quad \forall (x,t) \in Q_T, \\
v|_{t=0} = 0.
\end{aligned}
\]

The solution of problem (6) we seek in the subspace of such functions which are homogeneous at the initial moment of time: \( B = \{ v \in V : v|_{t=0} = 0 \} \).

Denoting the structure elements in (6) by
\[
A := \partial_t + a \partial^2_x, \quad \tilde{f} = -a\varphi^* + f \in C(Q_T),
\]
we obtain a problem for operator equation:
\[
\begin{aligned}
\text{for given operator } A : B \to C \text{ and element } \tilde{f} \in C, \\
\text{find element } v \in B \text{ such that } Av = \tilde{f}.
\end{aligned}
\]
is given a proof of the factorial convergence of the built numerical scheme. Multidimensional generalization of the numerical scheme is covered in sixth and seventh chapters. Numerical example is provided in eighth chapter.

3. NUMERICAL SCHEME AND UNIQUENESS OF DISCRETE SOLUTION

Let \( n_i \) denotes the count of nodes in domain \( \Omega \) and \( n_t \) denotes count of nodes in interval \([0, T]\). Set of nodes we denote \( Q_{T,h}. \)

For every variable we construct a set of Lagrange polynomials, which satisfy property \( l_j(x_i) = \delta_{ij} \) and \( l_j(t_i) = \delta_{ij} \), where \( \delta_{ij} \) denotes Kronecker symbol.

According to the Weierstrass approximation theorem the set of polynomials with real-valued coefficients is dense set in the space of continuous real-valued functions. Choosing \( l(t) = \left[ l_j(t) \right]_{j=1}^h \) we obtain system of polynomials without polynomial associated with initial moment of time. Its easy verifies that \( \forall j = 2, n_t, l_j(t) \big|_{t=0} = 0 \) and \( l(t) \in B \), moreover basis functions \( l(t) \otimes l(x) \in B \) are linearly independent, hence system of these functions create basis for approximation spaces \( B_h \subset B, C_n \subset C \). Thus, we seek the solution as a Lagrange interpolation in the following form

\[
v_h(x,t) = \sum_{j=1}^n \sum_{l=1}^n v_j f_j \left( x_l, t_i \right) = \bar{v} \left( l'(t) \otimes l(x) \right), \tag{8}\]

where \( h \) denotes the discretization parameter, \( j \) and \( j_s \) are indexes of nodes by corresponding variables, \( j \) denotes the unique number of the node \( j = (j_s - 1)n_l + j_l \) and \( \bar{v} \) denotes the set of values \( \bar{v} = \left( v_j \right)_{j=1}^n \).

Substitution (8) into equation (7) leads to \( Av_h = \bar{f} \) and further using of calculations yields

\[
(l'(t) \otimes l(x) - l(t) \otimes l'(x)) \otimes h = \bar{f}_h. \tag{9}\]

Taking \( i_l = 1, n_l \) and \( i_s = 2, n_s \) in (9) we obtain a system of linear algebraic equations

\[
\left( Z \otimes I_s - I_s \otimes Z \right) \bar{v} = \bar{f}_s(x_i, t_j), \quad i_s = 1, n_s, \quad i_l = 2, n_l.
\]

Denoting \( A_h = Z \otimes I_s - I_s \otimes Z \) and \( \bar{f}_h = \bar{f}(x_i, t_j), \quad i_s = 1, n_s, \quad i_l = 2, n_l \) we obtain discrete formulation of operator problem

\[
\begin{align*}
\text{for given operator} & \quad A_h : B_h \rightarrow C_h \text{ and element} \quad \bar{f}_h \in C_h, \\
\text{find element} & \quad v_h \in B_h \text{ such, that} \quad A_h v_h = \bar{f}_h,
\end{align*}
\]

where matrices of corresponding finite dimensional quasi representations have been built upon these rules

\[
Z_{ij} = l'_j(t_i), \quad Z_{ij} = \left( l'_j \right)(x_i), \quad I_{ij} = l_j(t_i), \quad I_{ij} = l_j(x_i).
\]

According to theorem determining the rank of finite dimensional quasi representations [13] we obtain

\[
\text{rank} \left( Z \right) = n_t - 1, \quad \text{rank} \left( I \right) = n_s - 1, \quad \text{rank} \left( Z \right) = n_t - 2, \quad \text{rank} \left( I \right) = n_s.
\]

Using property of tensor product we verify that
The rank of whole matrix $A_n$ remains an open question, and further lemmas give an answer to this question.

**Lemma 3.1.** Matrix $\left(Z^{-1} \otimes Z^2_{i}\right)$ is nilpotent.

*Proof.* Since finite dimensional quasi representation of operator $d^2/\text{dx}^2$ has the form $Z^{-1}_{i} = Z_{i}$ and matrix $Z^{-1}_{i}$ is nilpotent, hence $\exists m \in \mathbb{N}, \forall n \geq m: \left(Z^{-1}_{i}\right)^m = 0$.

The property of tensor product [5, 13] $\forall n \in \mathbb{N} \ (A \otimes B)^* = A^* \otimes B^*$ yields

$\exists m \in \mathbb{N}, \forall n \geq m: \left(Z^{-1}_{i} \otimes Z^2_{i}\right)^m = \left(Z^{-1}_{i}\right)^m \otimes \left(Z^2_{i}\right)^m = \left(Z^{-1}_{i}\right)^m \otimes \left(Z^2_{i}\right)^m = 0$, hence matrix $\left(Z^{-1}_{i} \otimes Z^2_{i}\right)$ is nilpotent.

**Lemma 3.2.** Matrix $\left(I_{i} \otimes I_{i} + a Z^{-1}_{i} \otimes Z^2_{i}\right)$ has an inverse matrix and its rank is $(n_i - 1) n_i$ i.e. has full rank.

*Proof.* Let us rewrite matrix $\left(I_{i} \otimes I_{i} + a Z^{-1}_{i} \otimes Z^2_{i}\right)$ as a formal series:

$\left(I_{i} \otimes I_{i} + a Z^{-1}_{i} \otimes Z^2_{i}\right) = a \sum_{m=0}^{\infty} (-1)^m \left(Z^{-1}_{i} \otimes Z^2_{i}\right)^m = a \sum_{m=0}^{\infty} (-1)^m \left(Z^{-1}_{i}\right)^m \otimes \left(Z^2_{i}\right)^m$.

Since matrices $Z^{-1}_{i}, \left(Z^{-1}_{i} \otimes Z^2_{i}\right)$ are nilpotent (Lemma 3.1) we obtain that the inverse matrix exists because of the existence of finite expansion

$\left(I_{i} \otimes I_{i} + a Z^{-1}_{i} \otimes Z^2_{i}\right) = a \sum_{m=0}^{\infty} (-1)^m \left(Z^{-1}_{i}\right)^m \otimes \left(Z^2_{i}\right)^m$.

However matrix has $n_i (n_i - 1)$ rows and columns and has an inverse matrix therefore it has full rank: $\text{rank} \left(I_{i} \otimes I_{i} - Z^{-1}_{i} \otimes K_{i}\right) = n_i (n_i - 1)$.

Using these lemmas we can prove the next theorem.

**Theorem 3.1.** The rank of finite dimensional quasi representation $A_n$ of operator $A$ has full rank and its rank is $n_i (n_i - 1)$ and there exists a unique solution of discrete problem (10).

*Proof.* Let us rewrite

$A_n = Z_i \otimes I_i + a I_i \otimes Z^2_{i} = \left(Z_i \otimes I_i + a Z^{-1}_{i} \otimes Z^2_{i}\right)$. However $\text{rank} \left(Z_i \otimes I_i\right) = n_i (n_i - 1)$ and due to lemma 3.2

$\text{rank} \left(I_i \otimes I_i + a Z^{-1}_{i} \otimes Z^2_{i}\right) = n_i (n_i - 1)$ using property for two square matrices $A, B$: $\text{rank} (AB) = \min \{\text{rank} (A), \text{rank} (B)\}$, yields

$\text{rank} \left(Z_i \otimes I_i + a I_i \otimes Z^2_{i}\right) = n_i (n_i - 1)$.

Since matrix has full rank then a unique solution of the problem (10) exists there.

4. APPROXIMATION PROPERTIES OF NUMERICAL SCHEME

According to construction of finite dimensional quasi representation of the operator it can be verified that $(Av - A_n v_i)_{\text{h} \to \text{m}} = (Av(M) - Av_i (M))_{\text{h} \to \text{m}}$, where $v_i$ denotes Lagrange interpolant and $M_i = \{x_i, t_i\}$ denotes node from $Q_{7, k}$.

Let the dimension of finite dimensional subspaces $B_n, C_n$ be $\dim B_n = \dim C_n = N_n$. In these spaces we can define the norm in a similar way as it was proposed in [5] namely
\[
\|v\|_h = \|w\|_h = \sqrt{\frac{1}{N_h} \sum_{j=1}^{N_h} v_j^2}.
\]

Assume that \( v \in W^{n,n_0} \equiv \{ v : Q_T \rightarrow R : D^s v \in L^s (Q_T), \forall s \leq n, n_0 \} \) which means that all possible derivatives till order \( n, n_0 \) are bounded.

The residual of Lagrange interpolation polynomial can be written in the following form:

\[
v(x,t) - v_j(x,t) = \frac{\omega_{n_j} (x)}{(n_j)!} \frac{\partial^n v(x,t)}{\partial x^n} + \frac{\omega_{n_j} (t)}{(n_j)!} \frac{\partial^n v(x,t)}{\partial t^n} - \frac{\omega_{n_j} (x) \omega_{n_j} (y)}{(n_j)!} \frac{\partial^n v(\xi_j, \eta_j)}{\partial x^n \partial t^n}.
\]

where

\[
\omega_{n_j} (x) = \prod_{i=1}^{n_j} (x - x_i), \omega_{n_j} (y) = \prod_{i=1}^{n_j} (y - y_i) \text{ and } \xi_j, \eta_j \in \Omega, \eta, \eta_i \in (0,T).
\]

**Theorem 4.1.** Finite dimensional quasi representation \( A_h \) of the operator \( A \) approximates the operator \( A \) on element \( v \in B \) and error estimation of approximation has the following form

\[
\|A v - A_h v\|_{C_e} \leq \ln(n_j) \left( \frac{1}{n_j - 1} \right)^{n-1} \left\| \frac{\partial^n v}{\partial x^n} \right\|_{\infty} + a \ln(n_j) \ln(n_j - 1) \left( \frac{1}{n_j - 1} \right)^{n-2} \left\| \frac{\partial^n v}{\partial t^n} \right\|_{\infty}.
\]

**Proof.** Since the norm of space \( C_e \) is vector norm then according to the construction of finite quasi representation \( A_h \) of operator \( A \) it can verified that

\[
(A v - A_h v) = (A(M) - A v_j (M))_{M \in M},
\]

where \( v_j \) denotes Lagrange interpolant and \( M_i = \{ x_i \}, t_i \) denotes node from \( Q_{T_n} \). Using the definition of the norm in space \( C_e \) after calculations we obtain

\[
\|A v - A_h v\|_{C_e} \leq \|A v - A v_j\|_{C_e} = \|v - v_j\|_{C_e}.
\]

Acting with operator \( A \) on residual of Lagrange polynomial we obtain

\[
A(v(x,t) - v_j(x,t)) \approx a - \frac{\omega_{n_j}^2 (x)}{(n_j)!} \frac{\partial^n v}{\partial x^n} + \frac{\omega_{n_j} (t)}{(n_j)!} \frac{\partial^n v}{\partial t^n}.
\]

Estimation

\[
|\omega_{n_j}^2 (x)| \leq \left( n_j \right) \ln(n_j) \left( \frac{1}{n_j - 1} \right)^{n-1}, |\omega_{n_j} (t)| \leq \left( n_j \right) \ln(n_j) \ln(n_j - 1) \left( \frac{1}{n_j - 1} \right)^{n-2}
\]

and \( v \in W^{n,n_0} (Q_T) \cap B \) yields

\[
\|A v - A v_j\| \leq \ln(n_j) \left( \frac{1}{n_j - 1} \right)^{n-1} \left\| \frac{\partial^n v}{\partial x^n} \right\|_{\infty} + a \ln(n_j) \ln(n_j - 1) \left( \frac{1}{n_j - 1} \right)^{n-2} \left\| \frac{\partial^n v}{\partial t^n} \right\|_{\infty},
\]

and finally (11) can be obtained.

5. CONVERGENCE AND ERROR ESTIMATIONS

According to the Kantorovich convergence theorem [8] of abstract approximation scheme \( \lim_{h \to 0} \|v - v_j\| = 0 \) holds if
1. there exists a unique solution of equation $Av = \tilde{f}$,
2. for all operators approximating $A_\alpha$ operator $A$ exist inverse bounded operators,
3. operator $A_\alpha$ approximates operator $A$ on element $v \in B$: $\lim_{h \to 0} \| A_\alpha v - A v \|_{c_1} = 0$.

The first requirement can be easily verified and the third requirement has been already satisfied in theorem 2, thus we should prove the second requirement.

**Theorem 5.1.** If finite dimensional quasi representation $A_\alpha$ of operator $A$ has a full rank and is the same as finite dimensional subspace $B_\alpha$ then the bounded inverse operator exists i.e.:

$$\forall A_\alpha, \exists M > 0, \exists A_\alpha^{-1}: \| A_\alpha^{-1} \| \leq M < +\infty.$$  \hspace{1cm} (12)

**Proof.** Although the norm satisfies the axiom of positivity, we obtain $\| A_\alpha v \|_{c_1} \geq 0$, $\forall v \in D(A_\alpha)$ and $\| A_\alpha v \|_{c_1} = 0 \iff A_\alpha v = 0$. Let $\| A_\alpha v \|_{c_1} = 0$, then $A_\alpha v = 0$. For $v \neq 0$, it is possible if $\det A_\alpha = 0$ and $\text{rank} A_\alpha < \dim B_\alpha$. However $A_\alpha$: $\text{rank} A_\alpha = \dim B_\alpha$ then $A_\alpha v = 0$ is possible if $v = 0$. Then $\forall v \in D(A_\alpha) \setminus \{0\}$: $\| A_\alpha v \|_{c_1} > 0$. Since values $\| A_\alpha v \|_{c_1} > 0$ and $\| v \|_{c_1} > 0$ are strictly positive for $\forall v \in D(A_\alpha) \setminus \{0\}$ then there exists such constant $\mu > 0$ such that $\| A_\alpha v \|_{c_1} \geq \mu \| v \|_{c_1}$.

According to the theorem of existence of bounded inverse operator [5] setting $M = \frac{1}{\mu} > 0$ we obtain (12).

**Theorem 5.2.** If $\lim_{h \to 0} \| \tilde{f} - \tilde{f}_h \|_{c_1} = 0$ holds and conditions of theorems 3.1, 4.1, 5.1 are satisfied then $\lim_{h \to 0} \| u - u_h \|_{\mathfrak{B}^0} = 0$ i.e. numerical scheme (10) is convergent and

$$\| u - u_h \|_{\mathfrak{B}^0} \leq M \left( \ln(n_\alpha) \left( \frac{1}{n_\alpha - 1} \right)^{1 - \frac{1}{\alpha}} \frac{\| v \|_{c_1}}{\| v \|_{c_1}} + \| a_n \|_{c_1} \left( \sum_{k=0}^{n_\alpha - 1} \ln(n_\alpha - k) \right) \frac{1}{n_\alpha - 1} \left( \frac{\| v \|_{c_1}}{\| v \|_{c_1}} \right)^{\alpha - 1} \right)$$  \hspace{1cm} (13)

**Proof.** Let us consider $\| v - v_h \|_{c_1}$.

$$\| v - v_h \|_{c_1} = A_\alpha^{-1} A_\alpha (v - v_h) \leq \| A_\alpha^{-1} \| A_\alpha (v - v_h) \|_{c_1}.$$  

Since the inverse operator is bounded (12) then

$$\| A_\alpha^{-1} \| A_\alpha (v - v_h) \|_{c_1} \leq M \| A_\alpha (v - v_h) \|_{c_1}.$$  

Let us estimate $\| A_\alpha (v - v_h) \|_{c_1}$:

$$\| A_\alpha (v - v_h) \|_{c_1} = \| A_\alpha (v - Av + Av - Av_h) \|_{c_1} \leq \| A_\alpha (v - Av) \|_{c_1} + \| Av - Av_h \|_{c_1}.$$  

Since $Av = \tilde{f}$ and $A_\alpha v = \tilde{f}_h$, we obtain

$$\| A_\alpha (v - v_h) \|_{c_1} \leq \| A_\alpha (v - Av) \|_{c_1} + \| \tilde{f} - \tilde{f}_h \|_{c_1}.$$  

Finally we obtain
\[ \|v - v_h\|_{b_h} \leq M \left( \|A_v v - Av\|_{C_t} + \|\vec{f} - \vec{f}_h\|_{C_t} \right) \]

and

\[ \lim_{h \to 0} \|v - v_h\|_{b_h} \leq M \left( \lim_{h \to 0} \|A_v v - Av\|_{C_t} + \lim_{h \to 0} \|\vec{f} - \vec{f}_h\|_{C_t} \right) = 0, \]

thus \( \lim_{h \to 0} \|v - v_h\|_{b_h} = 0 \).

Substitution leading to homogeneous initial condition \( u = v + \varphi \) implies \( u_h = v_h + \varphi \) which yields the estimation \( \|u - u_h\| = \|v + \varphi - v_h - \varphi\| = \|v - v_h\| \). However \( \vec{f}_h \) is Lagrange approximation of \( \vec{f} \), error estimation of \( \|\vec{f} - \vec{f}_h\|_{b_h} \) has the following form:

\[ \|\vec{f} - \vec{f}_h\|_{b_h} \leq \left( \frac{1}{n_j - 1} \right)^n \frac{\|\varphi\|_{C_t}}{2\pi^n} + \left( \frac{1}{n_s - 1} \right)^n \frac{\|\varphi\|_{C_t}}{2\pi^n}. \]

Since (11) and \( \left( \frac{1}{n_j - 1} \right)^n \) tends to zero faster than \( \left( \frac{1}{n_j - 1} \right)^{n-1} \) when \( n_j \to \infty \), then neglecting terms with \( \left( \frac{1}{n_j - 1} \right)^n \) and \( \left( \frac{1}{n_s - 1} \right)^n \) yields to error estimation (13).

6. THE CAUCHY PROBLEM FOR BACKWARD HEAT EQUATION IN SEVERAL DIMENSIONS

The results obtained in previous sections can be generalized in natural way for a multidimensional case. Let us consider \( q \)-dimensional bounded domain \( \Omega := (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_q, b_q) \subset \mathbb{R}^q \), time limit \( T < +\infty \) and cylinder \( Q_T = \Omega \times (0, T] \).

Let \( \text{diam} \) denote the length of the range \((a, b)\). We assume, that linear differential operator \( K \) is formal polynomial of elements from Lie algebra \( \Theta \{ \partial / \partial x_i, \partial / \partial x_j \} \) [3, 10]. Let \( a_i \) denotes coefficient standing by second derivative by variable \( x_i \).

Let operator \( K \) has the following representation

\[ K = -\sum_{i=1}^{q} a_i \frac{\partial^2}{\partial x_i^2}, \quad a_i > 0, \quad i = 1, q \]

and let us consider Banach spaces \( V = C^{2,1}_{n_1,\ldots,n_q}(Q_T) \cap C(\overline{Q_T}), \quad C = C(Q_T) \) and formulation of the Cauchy problem with linear backward heat equation [20] is given below

\[
\text{find function } u = u(x_1, \ldots, x_q, t) \in V \text{ such that, that } \]

\[
u_t = Ku + f, \quad \forall (x_1, \ldots, x_q, t) \in Q_T, \quad (14)\]

\[
u|_{t=0} = \varphi \in V, \]

where \( \varphi = \varphi(x_1, \ldots, x_q) \in V \) denotes initial conditions and \( f = f(x_1, \ldots, x_q, t) \in C \) represents internal sources and

According to [3, 19] we introduce substitution
$$u(x_1,\ldots,x_q,t) = v(x_1,\ldots,x_q,t) + \varphi(x_1,\ldots,x_q)$$
into (14) which leads to considering an auxiliary Cauchy problem with homogeneous initial condition:

$$\begin{cases}
\text{find function } v = v(x_1,\ldots,x_q,t) \in V \text{ such that } \\
v_i = Kv + K\varphi + f, \ \forall (x_1,\ldots,x_q,t) \in Q_T, \\
v|_{t=0} = 0.
\end{cases} \quad (15)$$

The solution of problem (15) we seek in the subspace of such functions which are homogeneous at initial moment of time: $B = \{ v \in V : v|_{t=0} = 0 \}$.

Denoting structure elements in (15) by $A := \partial / \partial t - K, \ \tilde{f} = K\varphi + f \in C(Q_T)$
we obtain the problem for operator equation:

$$\begin{cases}
\text{for given operator } A : B \to C \text{ and element } \tilde{f} \in C \\
\text{find element } v \in B \text{ such that } Av = \tilde{f}.
\end{cases} \quad (16)$$

The Cauchy problem has been reduced into problem for operator equation. This operator equation we solve by means of Generalized Method of Lie algebraic discrete approximations.

7. APPROXIMATION PROPERTIES AND CONVERGENCE IN MULTIDIMENSIONAL CASE

The numerical scheme for problem (16) is built using dimensional Lagrange interpolation. Let $n_i$ denote the count of nodes by variable $x_i$.

Discrete problem is formulated below

$$\begin{cases}
\text{for given operator } A_n : B_n \to C_n \text{ and element } \tilde{f}_h \in C_h \\
\text{find element } v_h \in B_n \text{ such that } A_nv_h = \tilde{f}_h.
\end{cases} \quad (17)$$

**Theorem 7.1.** The rank of finite dimensional quasi representation $A_n$ of the operator $A$ has a full rank and its rank is $(n_1-1)\prod_{i=1}^q n_i$ and there exists a unique solution of the discrete problem (17).

**Proof.** Using property that finite dimensional quasi representation $K_n$ of operator $K$ is nilpotent matrix. Similarly, as in proof of theorem 3.1, we obtain that finite dimensional quasi representation has a full rank and hence a unique solution of discrete problem (17) exists there.

**Theorem 7.2.** Finite dimensional quasi representation $A_n$ approximates operator $A$ on element and error estimation of approximation has the following form

$$\left\| A v - A_n v \right\|_n \leq \prod_{i=1}^q \left( \frac{1}{n_i-1} \right)^{n_i-1} \left\| \frac{\partial^\nu v}{\partial x_i^\nu} \right\|_n +$$

$$+ \sum_{\nu=2}^p \left( \prod_{i=1}^q \left( \frac{1}{n_i-1} \right)^{n_i-1} \right)^{\nu-2} \left\| \frac{\partial^\nu v}{\partial x_i^\nu} \right\|_n.$$ 

(18)
Proof. Using formula of dimensional Lagrange interpolation and acting by operator $A$ on residual yields (18).

**Theorem 7.3.** If $\lim_{h \to 0} \| \tilde{f} - f \|_{C_{x}} = 0$ holds and conditions of theorems 5.1, 7.1, 7.2 are satisfied then $\lim_{h \to 0} \| u - u_{h} \|_{C_{x}} = 0$ and

$$\| u - u_{h} \|_{C_{x}} \leq M \left[ \sum_{i=1}^{n} \left| a_{i} \ln(n_{i}) \ln(n_{i} - 1) \left( \frac{1}{n_{i} - 1} \right)^{2} \right| \right]$$

(19)

Proof. According to the Kantorovich convergence theorem of abstract approximation scheme all requirements are satisfied, thus similarly to theorem 5.2 using inequality

$$\| u - u_{h} \|_{C_{x}} \leq \| v_{h} \|_{C_{x}} \leq M \| A_{y} - A_{h} \|_{C_{x}}$$

and neglecting terms $\left( \frac{1}{n_{i} - 1} \right)^{2}$ and $\left( \frac{1}{n_{i} - 1} \right)$ we obtain (19).

8. NUMERICAL EXAMPLE

We consider model problem with backward heat equation which is similar to model problem in [4, 12]

$$\begin{align*}
\text{find function } & u = u(x, t) \text{ such that: } \\
\frac{\partial u}{\partial t} & = -\frac{\partial^{2} u}{\partial x^{2}}, \forall(x, t) \in Q_{T}, \\
\left| u \right|_{t=0} & = \sin x,
\end{align*}$$

(8.1)

having the exact solution $u(x, t) = e^{t} \sin x$.

If the exact solution is known, we use the following rule for evaluating the rate of convergence:

$$p_{h} = \log_{2} \left( \frac{\| u - u_{h} \|_{C_{x}}}{\| u - u_{h/2} \|_{C_{x}}} \right).$$

If we get value $\| u - u_{h} \| = 0$ and $\| u - u_{h/2} \| = 0$, thus the value 0/0 is shown as NaN (not a number).

The model problem is investigated by means Lax–Wendroff scheme of finite differences method (FDM), method of Lie-algebraic discrete approximations (MLADA) and generalized method of Lie-algebraic discrete approximations (GMLADA). The solution of Cauchy problem with the system of differential equations was performed using Mathematica.

Let us denote $\Delta x = 1/(n_{x} - 1)$ – the step of discretization by space variable, $\Delta t = 1/(n_{t} - 1)$ – the step of discretization by time variable. If discretization steps by both variables are equal then we use $h = \Delta x = \Delta t$ for FDM and GMLADA. Nevertheless $h$
denotes the step of discretization by space variable, because time step is chosen automatically while solving the Cauchy problem with the system of differential equation by means of Mathematica software.

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Step $h$ & FDM & MLADA & GMLADA \\
\hline
$h = 1/2$ & 0.129611 & 0.256518 & 0.0844799 \\
$h = 1/4$ & 0.071353 & 0.0808159 & 0.020411 \\
$h = 1/8$ & 0.0380837 & 0.00404986 & 0.000689873 \\
$h = 1/16$ & $-$ & $-$ & 1.43891$\times 10^{-7}$ \\
\hline
\end{tabular}
\caption{Error estimations in $L^2(Q_T)$ space}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Step $h$ & FDM & MLADA & GMLADA \\
\hline
$h = 1/2$ & 0.420034 & 0.976364 & 0.36965 \\
$h = 1/4$ & 0.241512 & 0.356581 & 0.099300 \\
$h = 1/8$ & 0.130843 & 0.0212378 & 0.0038209 \\
$h = 1/16$ & $-$ & $-$ & 9.1009$\times 10^{-7}$ \\
\hline
\end{tabular}
\caption{Error estimations in $L^\infty(Q_{T,T})$ space}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Step $h$ & FDM & MLADA & GMLADA \\
\hline
$h = 1/2$ & 0.413269 & 0.984519 & 0.374173 \\
$h = 1/4$ & 0.232347 & 0.360836 & 0.104918 \\
$h = 1/8$ & 0.124427 & 0.0239794 & 0.00464834 \\
$h = 1/16$ & $-$ & $-$ & 1.49632$\times 10^{-6}$ \\
\hline
\end{tabular}
\caption{Error estimations in $W^{1,2}(Q_T)$ space}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Step $h$ & FDM & MLADA & GMLADA \\
\hline
$h = 1/2$ & 0.861099 & 1.66635 & 2.04926 \\
$h = 1/4$ & 0.905849 & 4.3187 & 4.88687 \\
$h = 1/8$ & $-$ & $-$ & 12.2271 \\
\hline
\end{tabular}
\caption{Rates of convergence in $L^2(Q_T)$ space}
\end{table}
Table 5
Rates of convergence in $L^\infty(Q_T)$ space

<table>
<thead>
<tr>
<th>Step $h$</th>
<th>FDM</th>
<th>MLADA</th>
<th>GMLADA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1/2$</td>
<td>0.798413</td>
<td>1.45319</td>
<td>1.89631</td>
</tr>
<tr>
<td>$h = 1/4$</td>
<td>0.884252</td>
<td>4.06952</td>
<td>4.69979</td>
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<tr>
<td>$h = 1/8$</td>
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<td>–</td>
<td>12.0356</td>
</tr>
</tbody>
</table>

Table 6
Rates of convergence in $W^{1,2}(Q_T)$ space

<table>
<thead>
<tr>
<th>Step $h$</th>
<th>FDM</th>
<th>MLADA</th>
<th>GMLADA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1/2$</td>
<td>0.830802</td>
<td>1.44808</td>
<td>1.83445</td>
</tr>
<tr>
<td>$h = 1/4$</td>
<td>0.900985</td>
<td>3.91147</td>
<td>4.4964</td>
</tr>
<tr>
<td>$h = 1/8$</td>
<td>–</td>
<td>–</td>
<td>11.6011</td>
</tr>
</tbody>
</table>

9. CONCLUSIONS

We present the application of the generalized method of Lie algebraic discrete approximations for solving the Cauchy problem for backward heat equation in this paper. The key finding of this research is the opportunity to provide a factorial rate of convergence by all variables in the equation, including time variable. The Cauchy problem for backward heat equation has been reduced to a system of linear algebraic equations.

There were compared different numerical schemes (finite difference method, classical method of Lie-algebraic discrete approximations and generalized method of Lie-algebraic discrete approximations) for solving the Cauchy problem for backward heat equation.

Substitution allows the rapid solving of the problem when initial data or internal sources are mutable but coefficients of differential operator in the problem remained constant. That was available by keeping in memory the inverse matrix and multiplying it on the vector which represents initial data and/or internal sources. Such substitution encapsulates the idea which in some way separates the data of the problem and the internal structure of the problem.

REFERENCES

Розглянуто обчислювальну схему для наближеного розв’язування задачі Коші для оберненого рівняння теплопровідності з використанням узагальненого методу Лі-алгебричних дискретних апроксимацій. Дискретизацію рівняння виконано за усіма змінними, що уможливає факторіальну збіжність за усіма змінними, які входять до рівняння у випадку побудови квазізображень диференціального оператора з використанням інтерполяції Лагранжа. Визначено ранг скінченновимірного оператора, а також з’ясовано його апроксимаційні властивості. Наведено оцінки похибок, доведено факторіальну швидкість збіжності.

Key words: узагальнений метод Лі-алгебричних дискретних апроксимацій, обернене рівняння теплопровідності, скінченновимірне квазізображення, поліном Лагранжа, факторіальна збіжність.