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SOLUTION OF ROTATIONAL SHELL–FLUID ACOUSTIC INTERACTION PROBLEMS BY FINITE ELEMENT METHOD

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A mathematical model of a problem of rotational shell – fluid acoustic interaction in case of axially symmetric load is constructed. The model is based on linear statements of Timoshenko–Mindlin shells that take into account pressing of a normal element and acoustic approximation of fluid equations. An initial-boundary value problem of interaction of two media is formulated in terms of shell elastic displacement vector and fluid velocity potential, a corresponding variational interaction problem is constructed. A constructive proof of correctness of interaction problem is accomplished by means of Galerkin semi-discretization and a priori energy estimates.

A projection-mesh scheme that provides Galerkin semi-discretization by spatial variables employing quadratic finite element method approximations and one-step recurrent time integration scheme is constructed for solution of the variational problem. Using assumptions regarding the approximation spaces, which are typical for finite element method, a priori estimates of semi-discretization convergence speed are constructed.

Investigation of numerical solutions for common oscillations of a cylindrical shell filled with water, caused by different load types is carried out.

Key words: Timoshenko–Mindlin shell, variational formulation of the problem, projection mesh scheme, Galerkin semidiscretization, apriori convergence speed estimate, one-step recurrent scheme.

1. INTRODUCTION

Modern mechanism construction sets advanced condition for modeling of dynamic processes of acoustic interaction of various physical and mechanical. Complexity and insufficient knowledge about the interaction problems gives sufficient room for researches connected with increase of mathematical models complexity in order to fully take interaction processes into account.

In papers [4, 5] the authors used linear Timoshenko-Mindlin shell theory and acoustic fluid approximation for modeling of shell–fluid acoustic interaction. A complete mathematical research including determination of variational interaction problem well-posednes conditions, construction and mathematical argumentation of projection-mesh scheme of solution of the variational problem. This paper is based upon linear Timoshenko–Mindlin shell theory formulations taking into account pressing of normal element, whose construction and investigation are presented in [6], and acoustic approximation of fluid equations.

An initial boundary value problem as well as corresponding variational problem of interaction of two media are formulated [2]. By means of Galerkin semidiscretization and a priori energy estimates, a constructive proof of well-posednes of variational interaction problem is carried out [2]. A projection-mesh scheme that provides Galerkin semi-discretization by spatial variables employing quadratic finite element method approximations and one-step recurrent time integration scheme is constructed [3]. The estimations of Galerkin semidiscretization convergence are obtained, the stability conditions and one-step recurrent time integration scheme convergence are established [3].

2. PROBLEM FORMULATION

Consider an isotropic revolution shell with constant thickness h, whose elastic properties are determined by Yung module E and Poisson's factor ν. Let the shell be referred by a cylindrical coordinate system (r,z,θ) so that Oz axis coincides with shell symmetry axis. In this case, mid-surface of the shell is uniquely defined by its meridian Γ_S, which is parametrically determined as $r = r(z) \quad \forall z \in [0,L]$. We assume that the shell is completely filled with viscous incompressible fluid with density $\rho_0 = const > 0$, sound propagation velocity $c = const > 0$, and Ω – meridian cut of this fluid volume. Assuming that an external load causes only axisymmetric displacements of the structure at any time moment $t \in [0,T], 0 < T < +\infty$, an initial boundary value problem is formulated. A detailed description of the initial boundary value problem is given in [2]. Consider a variational problem of revolution shell–fluid interaction

$$\begin{cases} given\ \psi_0 \in \Phi,\ \psi_1 \in H,\ s_0 \in S,\ s_1 \in G \\ \quad l \in L^2(0,T;\Phi'),\ \lambda \in L^2(0,T;S'); \\ find\ \ a\ \ pair\, p = (\psi,s) \in L^2(0,T;\Phi\times S)\ \ such\ \ that \\ m(\psi''(t),\varphi) + a(\psi(t),\varphi) - b(s'(t),\varphi) = \langle l(t),\varphi\rangle, \\ \mu(s''(t),g) + \eta(s(t),g) + b(g,\psi'(t)) = \langle \lambda(t),g\rangle, \\ m(\psi'(0)-\psi_1,\varphi) = 0,\ \ \mu(s'(0)-s_1,q) = 0, \qquad \forall\varphi\in\Phi \\ a(\psi(0)-\psi_0,\varphi) = 0, \quad \eta(s(0)-s_0,q) = 0, \qquad \forall g = (v,y,\xi,\zeta)\in S, \quad \forall t \in (0,T] \end{cases} \tag{1}$$

in corresponding functional spaces

$$\begin{cases} V = \{v \in H^1((0,L)) \mid v(0) = v(L) = 0\}, \quad Y = \{y \in H^1((0,L))\}, \\ \Xi = \{\xi \in H^1((0,L)) \mid \xi(0) = \xi(L) = 0\}, \qquad \mathrm{Z} = \{\zeta \in H^1((0,L))\ \}, \\ S = V\times Y\times \Xi \times \mathrm{Z}, \qquad\qquad G = [L^2((0,L))]^4, \\ \Phi = \{\varphi \in H^1(\Omega)\}, \quad H = L^2(\Omega), \qquad Q = S\times\Phi, \qquad X = G\times H. \end{cases}$$

Linear and bilinear forms above are determined as follows

$$
\begin{cases}
m(\psi,\varphi)=\int\limits_{\Omega}\frac{1}{\rho_0 c^2}\psi\varphi r dr dz, \qquad a(\psi,\varphi)=\int\limits_{\Omega}\frac{1}{\rho_0}\left(\frac{\partial\psi}{\partial r}\frac{\partial\varphi}{\partial r}+\frac{\partial\psi}{\partial z}\frac{\partial\varphi}{\partial z}\right) r dr dz,\\
\mu(s,g)=\int_0^L\left(\rho huv+\rho hwy+\frac{\rho h^3}{12}\gamma_1\xi+\frac{\rho h^3}{12}\gamma_3\zeta\right)A_1A_2dz,\\
\eta(s,g)=\int_0^L\left\{\sum_{j=1}^{2}\left[N_{jj}(s)\varepsilon_{jj}(g)+M_{jj}(s)\kappa_{jj}(g)\right]+\right.\\
\qquad\left.+N_{33}(s)\varepsilon_{33}(g)+N_{13}(s)\varepsilon_{13}(g)+M_{13}(s)\kappa_{13}(g)\right\}A_1A_2dz,\\
b(s,\varphi)=\int_0^L\left(w\varphi+\frac{h}{2}\gamma_3\varphi\right)A_1A_2dz, \qquad \forall\psi,\varphi\in\Phi,\quad \forall s,g\in S,\\
\langle l,\varphi\rangle=\int\limits_{\Omega}f_0\varphi r dr dz, \qquad <\lambda,g>=\int_0^L\left(P_1v+P_3y+m_1\xi+\frac{h}{2}m_3\zeta\right)A_1A_2dz.
\end{cases} \tag{2}
$$

Here $s=(u,w,\gamma_1,\gamma_3)$ – vector of shell mid-surface displacements, ψ – fluid velocity potential, $\sigma=(N_{11},N_{22},N_{33},N_{13},M_{11},M_{22},M_{13})^T$ – column matrix of non-zero components of forces and moments tensor, $E=(\varepsilon_{11},\varepsilon_{22},\varepsilon_{33},\varepsilon_{13},\kappa_{11},\kappa_{22},\kappa_{13})^T$ – column matrix of non-zero components of shell deformation tensor, $P=(P_1,P_3,m_1,m_3)^T$ – vector of shell external load. A given function $f_0=f_0(r,z,t)$ describes sound sources distributed in the fluid.

Note that continuous bilinear forms (2) have the following properties:

$a(.,.):\Phi\times\Phi\to R$ – symmetrical and Φ -elliptic,

$m(.,.):H\times H\to R$ – symmetrical and H-elliptic,

$\mu(.,.):G\times G\to R$ – symmetrical and G- elliptic,

thus, they define dot products in spaces Φ, H and G correspondingly. This makes it possible to introduce the norms

$$\|\varphi\|_\Phi=\sqrt{a(\varphi,\varphi)},\ \ \|\varphi\|_H=\sqrt{m(\varphi,\varphi)},\ \ \|g\|_G=\sqrt{\mu(g,g)},\ \ \forall\varphi\in\Phi,\ \ \forall g\in G.$$

Bilinear form $\eta(.,.):S\times S\to R$ is symmetrical. [6] shows that $\eta(.,.):S\times S\to R$ is S- elliptic, i.e. these exists $\eta_0=const>0$ such that

$$\eta(g,g)\ge\eta_0\|g\|_S^2 \quad \forall g\in S.$$

In this case bilinear from $\eta(.,.)$ defines a dot product in space S as well as norm

$$\|g\|_S=\sqrt{\eta(g,g)} \qquad \forall g\in S.$$

Let us also introduce energy norms

$$\|\varphi(t)\|_F=\sqrt{\|\varphi(t)\|_\Phi^2+\|\varphi'(t)\|_H^2},\quad \|g(t)\|_\Sigma=\sqrt{\|g(t)\|_S^2+\|g'(t)\|_G^2}.$$

Theorem (on well-posedness of the variational problem). There exists a unique solution $p=(\psi,s)$ of variational problem of acoustic interaction (1) such that

$p = (\psi, s) \in L^{\infty}(0,T;\Phi \times S),\ p' = (\psi', s') \in L^{\infty}(0,T;H \times G),\ p'' = (\psi'', s'') \in L^{\infty}(0,T;\Phi' \times S')$

and at the same time, the solution continuously depends upon initial values and right sides, i.e. there exists $C = const > 0$ such that

$$\|\psi(t)\|_F^2 + \|s(t)\|_\Sigma^2 = \|\psi(t)\|_\Phi^2 + \|\psi'(t)\|_H^2 + \|s(t)\|_S^2 + \|s'(t)\|_G^2 \le$$
$$\le C\left\{\|\psi_0\|_\Phi^2 + \|\psi_1\|_H^2 + \|s_0\|_S^2 + \|s_1\|_G^2 + \int_0^t \left(\|l(\tau)\|_{\Phi'}^2 + \|\lambda(\tau)\|_{S'}^2\right)d\tau\right\}.$$

Here Φ' and S' – spaces of continuous linear functionals determined on Φ and S correspondingly. The theorem is proved constructively by means of Galerkin semidiscretization and a priori energy estimates [2].

Note that conditions of variational problem (1) well-posing guarantee successful construction of projection-mesh schemes for its solution. This approach employing finite element method on spatial variables and one-step recurrent time integration schemes is based on results of [3, 1].

3. PROJECTION-MESH SCHEME

Numerical procedure of solution of variational problem (1) is based upon a projection-mesh scheme, whose first stage assumes Galerkin semidiscretization on spatial variables using finite element method approximations. This allows to define uniquely semidiscrete approximations as decompositions by basis functions $\varphi_1, \varphi_2, ..., \varphi_K$ and $g_1, g_2, ..., g_N$ in finite dimensional spaces $\{\Phi_h\} \subset \Phi$ and $\{S_h\} \subset S$ correspondingly. Then a semidiscrete approximation $(\psi_h(t), s_h(t))$ of the solution of a semidiscrete variational problem (1) is presented as follows

$$(\psi_h(t), s_h(t)) = \left(\sum_{j=1}^{K} \psi_j(t)\varphi_j, \sum_{k=1}^{N} s_k(t) g_k\right) \tag{4}$$

with factors $\{\psi_j(t)\}_{j=1}^K$ and $\{s_k(t)\}_{k=1}^N$., which are unknown at the moment.

A one-step recurrent scheme [4, 3], which uses quadratic approximation of the solution on each integration step and allows integration using a variable time step, is employed for time discretization of the variational problem. Employing of matrix notations allows the following algebraic presentation of the projection-mesh scheme:

$$\begin{bmatrix} \mathbf{M_F} + \frac{1}{2}\Delta t^2 \beta \mathbf{A_F} & -\frac{1}{2}\Delta t \mathbf{B} \\ \frac{1}{2}\Delta t \mathbf{B}^{\mathrm{T}} & \mathbf{M_S} + \frac{1}{2}\Delta t^2 \beta \mathbf{A_S} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}^{j+\frac{1}{2}} \\ \mathbf{G}^{j+\frac{1}{2}} \end{bmatrix} = -\frac{1}{2}\Delta t \begin{bmatrix} \mathbf{A_F} & \mathbf{0} \\ \mathbf{0} & \mathbf{A_S} \end{bmatrix} \begin{bmatrix} \mathbf{\Psi}^j \\ \mathbf{S}^j \end{bmatrix} +$$
$$+\frac{1}{2}\Delta t \begin{bmatrix} \mathbf{L}_j \\ \Lambda_j \end{bmatrix} + \begin{bmatrix} \mathbf{M_F} + \frac{1}{2}\Delta t^2 (\beta - \theta) \mathbf{A_F} & \mathbf{0} \\ \mathbf{0} & \mathbf{M_S} + \frac{1}{2}\Delta t^2 (\beta - \theta) \mathbf{A_S} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}^j \\ \mathbf{G}^j \end{bmatrix} \tag{5}$$
$$\begin{bmatrix} \mathbf{\Psi}^{j+1} \\ \mathbf{S}^{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}^j \\ \mathbf{S}^j \end{bmatrix} + \Delta t \begin{bmatrix} \mathbf{\Phi}^{j+\frac{1}{2}} \\ \mathbf{G}^{j+\frac{1}{2}} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{\Phi}^{j+1} \\ \mathbf{G}^{j+1} \end{bmatrix} = 2\begin{bmatrix} \mathbf{\Phi}^{j+\frac{1}{2}} \\ \mathbf{G}^{j+\frac{1}{2}} \end{bmatrix} - \begin{bmatrix} \mathbf{\Phi}^j \\ \mathbf{G}^j \end{bmatrix} \quad j = 0,1,..N_T$$

Note that matrices $\mathbf{M_F}, \mathbf{A_F}, \mathbf{M_S}, \mathbf{A_S}$ are positively defined [6]. Thus, two first equations of the projection-mesh scheme (3) have a unique solution $\left(\Phi^{j+1/2}, G^{j+1/2}\right)$, that allows to compute a vector pair $\left(\Phi^{j+1}, G^{j+1}\right)$, which is necessary for going to the next integration step, from the next two equations. Thus, recurrent scheme (3) allows to find consequently $\left(\Phi^{j+1}, G^{j+1}\right)$, $j = 0,1,...$ by means of consequent solution of linear algebraic equation systems with positively defined matrix.

Recurrent scheme (3) allows to match initial conditions exactly, and appropriate choice of factors $\Delta t, \beta, \theta$ of the scheme provides computational stability and accuracy [3]. A detailed analysis of projection-mesh scheme (3) regarding conditions of its stability, convergence, and construction of a priori estimates is carried out in [4, 3].

4. NUMERICAL EXAMPLE

Consider an aluminum ($E = 7 \cdot 10^{10}$ Pa, $\rho = 2.7 \cdot 10^3$ kg/m^3, $\nu = 0.3$) cylindrical shell with radius $R = 1.5$ m, height $L = 3$ m, thickness $h = 0.03$ m. Let the edge of the shell be firmly fixed, and the bottom firmly fixed in a massive block (see Fig.1). Computation are carried out for the case when the shell is filled with water ($\rho_0 = 1000$ kg/m^3, $c = 1500$ m/s). Assume that free surface of the fluid is limited by an absolutely rigid piston Γ_V. Hydroshock wave is caused by impulse $Q = Q(t)$, applied to the surface of the rigid piston. Impulse shape is shown on Fig.2. Here $T_1 = 0.002$ time of load action, $Q_0 = 1.5\,H$.

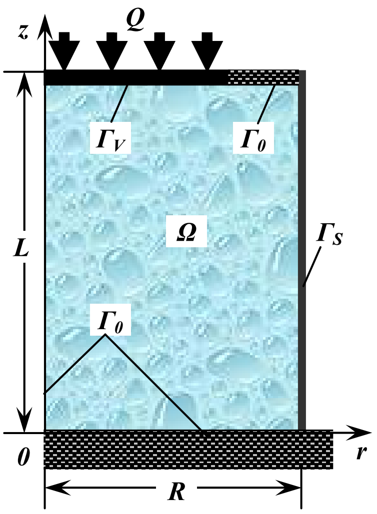

Fig.1. Construction profile.

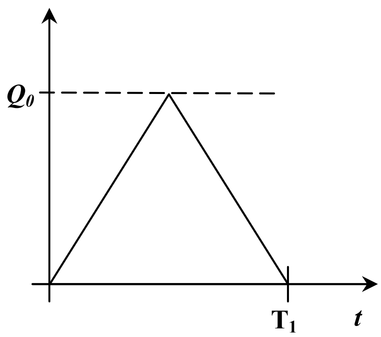

Fig. 2. Impulse shape.

Computations are carried out on an even mesh of the meridian section of the fluid, which consists of 900 rectangular finite elements (90×10) with quadratic approximation of the solution upon each finite element. Time integration was carried out by means of the one-step recurrent schemes with parameters $\beta = 0.51$, $\theta = 0.51$. Fixing the spatial variable step, Courant condition was applied to the time integration step $\Delta t \le h / 2c_1$, where c_1 – sound propagation velocity in the shell material. In order that a mesh of 90 finite elements along the meridian complies with this condition, we have to select time integration step of $\Delta t \le 0.1875 \cdot 10^{-5}$ c.

Fig. 3 shows distribution of the pressure in the fluid at different time moments. As one can see, presence of the shell significantly influences the shape of the hydroshock wave in liquid filler. One can see reduction of the pressure on the wave

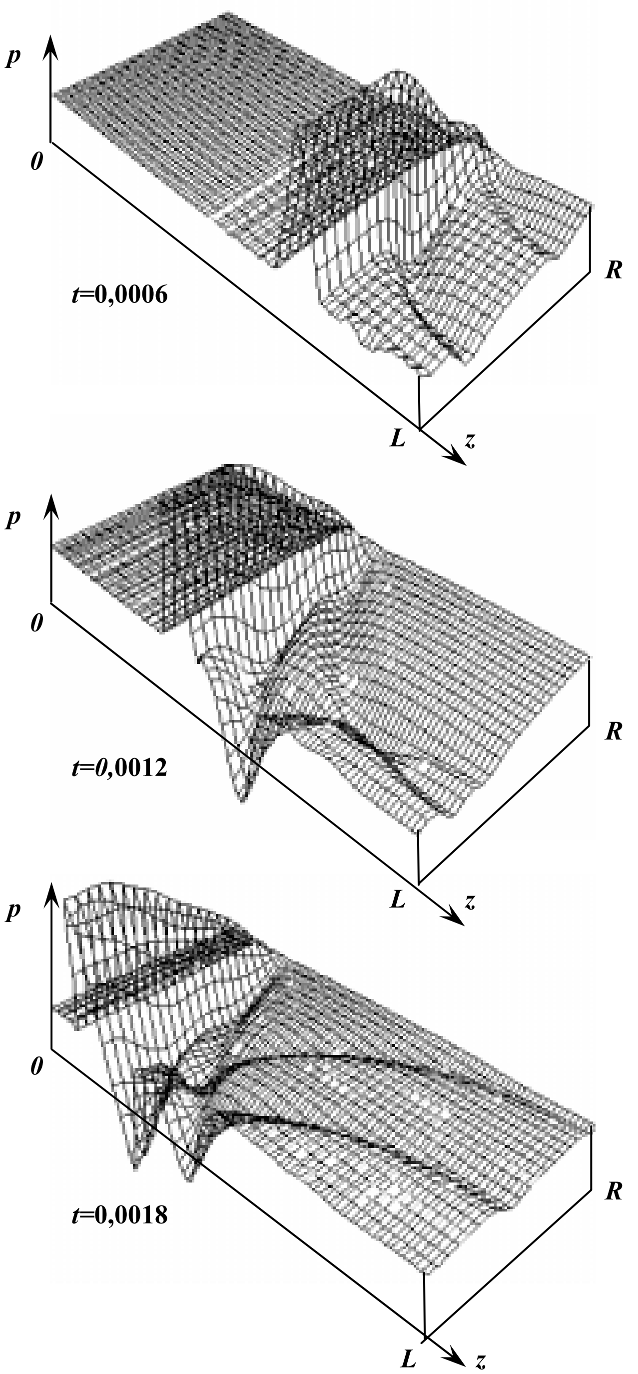

Fig. 3. Distribution of the pressure in the fluid at different time moments.

edge on approach to the surface of contact with shell. Behind the front of the hock wave, a rarefaction wave, which is caused by shell's response to fluid motion, appears. Explicitly seen rarefaction wave front (t=0.0009, t=0.0018) moves along the radial variable from the shell to the structure revolution axis. A point of an axis, which is on the edge of the rarefaction wave, is the point of appearance of a new rarefaction wave, which propagates from the revolution axis to the shell. In addition to this, in area of fluid contact with shell one can see perturbation of the pressure, which spread with velocity that is twice grater than speed of sound velocity in the fluid. For an aluminum shell, velocity of spread of the longitudinal wave is $c_1 = 5{,}34 \cdot 10^3$ m/s, cross wave – $c_2 = 3{,}16 \cdot 10^3$ m/s. Thus, perturbations in the fluid are caused by spreading of a cross wave in the shell.

5. CONSEQUENCE

Thus, wave process in the fluid causes shell displacements, which also influence pressure distribution in fluid. Moreover, under such load type, cross waves have significant influence on pressure distribution in fluid. Also, we would like to mention that employing of improved shell theory allows improved modeling of longitudinal waves in the shell.

LITERATURE

1. *Бернакевич І.Є.* Вплив циліндричної оболонки Тимошенка–Міндліна на поширення ударної хвилі в рідкому наповнювачі // Вісн. Львів. ун-ту. Сер. прикл. матем. та інформ. 2004. Вип. 9. С. 119–125.
2. *Бернакевич І.Є., Вагін П.П., Шинкаренко Г.А.* Математична модель акустичної взаємодії оболонки з рідиною: І. Формулювання і розв'язуваність варіаційних задач // Матем. методи та фіз.-мех. поля. 2002. Т. 45. № 2. С. 75–80.
3. *Бернакевич І.Є., Вагін П.П., Шинкаренко Г.А.* Математична модель акустичної взаємодії оболонки з рідиною: ІІ. Проекційно-сіткові апроксимації та їхня збіжність // Матем. методи та фіз.-мех. поля. 2004. Т. 47. № 3. С. 37–44.
4. *Бернакевич І.Є., Шинкаренко Г.А.* Чисельне моделювання акустичної взаємодії оболонок з рідиною. І. Формулювання і розв'язуваність варіаційних задач // Вісн. Львів. ун-ту. Сер. мех.-мат. 1995. Вип. 41. С. 7–14.
5. *Бернакевич І.Є., Шинкаренко Г.А.* Чисельне моделювання акустичної взаємодії оболонок з рідиною. ІІ. Проекційно-сіткові апроксимації та їхня збіжність // Вісн. Львів. ун- ту. Сер. мех.-мат. 1995. Вип. 42. С. 12–16.
6. *Вагін П.П., Іванова Н.В., Шинкаренко Г.А.* Аналіз зсувних оболонок: постановка та коректність варіаційних задач динаміки // Матем. студії. 1998. Т. 10. № 2. С. 188–198.

РОЗВ'ЯЗУВАННЯ ЗАДАЧ АКУСТИЧНОЇ ВЗАЄМОДІЇ ОБОЛОНОК ОБЕРТАННЯ З РІДИНОЮ МЕТОДОМ СКІНЧЕННИХ ЕЛЕМЕНТІВ

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Побудовано математичну модель задачі акустичної взаємодії оболонки обертання з рідиною у випадку осесиметричного навантаження. За основу моделі взято лінійні співвідношення оболонок Тимошенка–Міндліна з урахуванням обтиску нормального елемента й акустичне наближення рівнянь рідини. Сформульовано початково-крайову задачу взаємодії двох середовищ у термінах вектора пружних зміщень оболонки та потенціалу швидкостей рідини, побудовано відповідну їй варіаційну задачу взаємодії. За допомогою напівдискретизації Гальоркіна та апріорних енергетичних оцінок виконано конструктивне доведення коректності варіаційної задачі взаємодії.

Для розв'язування варіаційної задачі взаємодії побудовано проекційно-сіткову схему, яка передбачає напівдискретизацію Гальоркіна за просторовими змінними з використанням квадратичних апроксимацій методу скінченних елементів та однокрокову рекурентну схему інтегрування в часі. За типових для методу скінченних елементів припущень відносно просторів апроксимацій побудовано апріорні оцінки швидкості збіжності напівдискретних апроксимацій. Для рекурентної схеми інтегрування в часі визначено умови стійкості та побудовано апріорні оцінки швидкості збіжності.

Досліджено отримані числові розв'язки для циліндричної оболонки, заповненої рідиною, за їхнього сумісного коливання, спричиненого різними типами навантаження.

Ключові слова: оболонка Тимошенка–Міндліна, початково-крайова задача, варіаційна задача, проекційно-сіткова схема, апроксимація Гальоркіна, однокрокова рекурентна схема.

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